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On-off convection: Noise-induced intermittency near the convection threshold

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A phenomenological nonlinear stochastic model of intermittency experimentally observed by Behn, Lange, and John [Phys. Rev. E **58**, 2047 (1998)] in the electrohydrodynamic convection in nematics under dichotomous noise is proposed. This has the structure of the two-dimensional Swift-Hohenberg equation for local convection variable with fluctuating threshold. Numerical integration of the model equation shows intermittent emergence of convective pattern. Its statistics are found to obey those known, so far, for on-off intermittency. In the course of time, although the pattern intensity changes intermittently, no evident pattern change is observed. Adding additive noise, we observe an intermittent change of convective pattern.

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I. INTRODUCTION

Intermittency is a ubiquitous phenomenon in nonlinear dynamics, and is characterized by abrupt insertions of spatially or temporally highly random evolutions. The most famous one is known from the small-scale dynamics in hydrodynamic turbulence [1]. In dynamical systems, several types of intermittent evolution of dynamical variables have been observed in association with the destruction of limit cycle oscillations [2,3]. In 1985, on the other hand, an intermittency different from them was first reported when a particular chaotic motion, i.e., synchronized chaos in a coupled chaotic oscillator system, undergoes the instability as the coupling constant is changed [4]. This intermittency has quite interesting statistical laws [5], and is now known as on-off intermittency [6–8]. On-off intermittency has nowadays been observed in many fields of dynamical systems, particularly, with small degrees of freedom [7]. Furthermore very recently, the intermittency in systems with large degrees of freedom has been reported [8].

Recently, Behn, Lange, and John (BLJ) [9] developed the theory of the electrohydrodynamic convection in nematic liquid crystal system subject to the spatially uniform dichotomous noise without any temporally periodic field, and concluded the possibility of the existence of the instability leading to the onset of the electrohydrodynamic convection (EC) as the amplitude of the noise is increased. Conventionally, the survey of dynamics of the EC in the liquid-crystal system has been carried out under the application of a temporally periodic electric field [10]. In this sense, BLJ's prediction of the existence of the transition was quite unexpected. In addition, they predicted that the instability of planar alignment of directors may cause on-off intermittency. Very recently, John, Stannarius, and Behn [11] experimentally proved the existence of the transition, and furthermore verified the intermittency observed after the transition shows a signal of on-off intermittency, observing the laminar duration distribution, where the laminar state implies the planar alignment of directors.

It is known that the on-off intermittency has three characteristic statistics [5]: (i) the probability density $P(\rho)$ for $\rho(t)$, the magnitude of the deviation from the particular chaotic submanifold, obeys the asymptotic law $P(\rho)\propto\rho^{-1+\eta}$ with a small positive value η, (ii) the spectral intensity of the time series $\{\rho(t)\}$ exhibits a power law $\omega^{-(1/2)}$ in a low-frequency region, and (iii) given an appropriately small threshold $\rho_{\rm th}$, the probability density $Q(\tau)$ for the laminar duration τ takes an asymptotic form $Q(\tau)\propto\tau^{-(3/2)}$ in a certain wide range of τ [6]. The first two asymptotic laws are explained by solving a nonlinear multiplicative noise model for the time evolution of $\rho(t)$, and the third law is derived by the theory of the first passage time problem of Brownian motion, which is simply derived by dropping out the nonlinear term in the multiplicative noise model. Furthermore, according to the multiplicative noise model, the exponent η is obtained as

$$\eta=\frac{\lambda}{\Gamma}, \tag{1}$$

where $\lambda(>0)$ represents the deviation of the external control parameter from its critical value and Γ is the intensity of the modulational noise of the so-called transverse expansion rate. Equation (1) explains numerical results for several models quite well.

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The main aim of the present paper is to report a study associated with the intermittent onset of the convective pattern by utilizing a stochastic, dynamical model of EC subject to external noise. However, for this aim, we do not intend to describe the fundamental equation of motion from the basic electrohydrodynamic equations of motion, but we study with a phenomenological equation of motion, constructed by extending the basic equation in the convective problem in a simple neutral fluid.

The paper is prepared as follows. In Sec. II we propose a phenomenological equation of motion to imitate the BLJ instability. This has the same structure as the Swift-Hohenberg equation with the modulational threshold. A few characteristics of the model equation are discussed. The results of its numerical integration are given in Sec. III. It is shown that the instability of the quiescent state (planar alignment in the situation of BLJ) leads to the onset of the on-off intermittent generation of a convective pattern. Furthermore, in connection with the temporal change of the pattern form, the effect of additive noise is studied. We give concluding remarks in Sec. IV.

II. PHENOMENOLOGICAL MODEL OF ELECTROHYRODYNAMIC CONVECTION UNDER MULTIPLICATIVE NOISE

The convection problem in a simple, neutral fluid has been extensively studied both experimentally and theoretically [12–15]. Near the convection threshold, there appear two kinds of modes: critical and noncritical. The former is directly relevant to the formation of a convective pattern and the latter is stably slaved to the critical mode. Adiabatically eliminating noncritical modes, Swift and Hohenberg (SH) derived the amplitude equation

$$\frac{\partial w(\mathbf{r},t)}{\partial t}=[\lambda-(\boldsymbol{\nabla}^2+k_0^2)^2]w-w^3 \tag{2}$$

near the onset of convective pattern [13–15]. Here $w(\mathbf{r},t)$ is the vertical component of macroscopic local velocity of fluid at position $\mathbf{r}$ and time t, λ is the deviation of the Rayleigh number R from its critical value R_c, i.e., $\lambda=R-R_c$, k_0 is the most unstable wave number and $\boldsymbol{\nabla}^2$ the two-dimensional (2D) Laplacian. For $\lambda<0$, all modes for any wave number $\mathbf{k}$ are linearly stable, and therefore the conductive solution $w=0$ for any position $\mathbf{r}$ is stable. If, on the other hand, λ takes a small positive value, the modes with wave numbers near k_0 become unstable, which eventually produces the roll type convective pattern with the wave number k_0.

In the present paper, we phenomenologically extend the SH equation (2) noting that the applied external force in the BLJ case is a spatially uniform noise. We start with the Swift-Hohenberg equation with random modulation in the threshold term, i.e.,

$$\frac{\partial w(\mathbf{r},t)}{\partial t}=[\lambda+f(t)-(\boldsymbol{\nabla}^2+k_0^2)^2]w-w^3. \tag{3}$$

Here $w(\mathbf{r},t)$ is the amplitude of the local convection pattern, i.e., describes the amplitude of the space-charge density with a fundamental wave number or the gradient of the local angle between director and the electrode plates in the experimental situation of BLJ. The $f(t)$ is the applied spatially uniform modulation noise and is assumed to have a vanishing mean and the magnitude Γ_f,

$$\Gamma_f=\int_0^\infty\langle f(t)f(0)\rangle dt, \tag{4}$$

where the angular brackets stand for the ensemble average. It should be noted that the assumption that the applied external field is spatially uniform follows the theoretical—and the experimental—conditions in Refs. [9,11]. The model equation (3) always has a quiescent state $w=0$, which corresponds to the complete planar alignment of director to the electrodes. The linear stability of the conductive (planar alignment) state is examined with the $\mathbf{k}$-mode growth rate,

$$\lambda_{\mathbf{k}}=\lambda-(k^2-k_0^2)^2 \quad (k=|\mathbf{k}|). \tag{5}$$

If $\lambda<0$, there exists no unstable mode, and the spatial pattern eventually decays into the planar state. On the other hand, the planar state is unstable for modes with wave numbers around $k=k_0$ for $\lambda\geqslant 0$. As numerically shown later, this situation leads to the onset of the intermittent convective pattern.

Before carrying out the numerical integration of Eq. (3), let us comment on the characteristics of the phenomenological equation (3). In the one-dimensional (1D) system, by putting $w=\psi e^{ik_0x}+$c.c. with the complex amplitude ψ, which has a weak x dependence, Eq. (3) reduces to

$$\frac{\partial\psi(x,t)}{\partial t}=\left[\lambda+f(t)+4k_0^2\frac{\partial^2}{\partial x^2}\right]\psi-3|\psi|^2\psi. \tag{6}$$

Here we have put $w^3\approx 3|\psi|^2\psi e^{ik_0x}+$c.c. by retaining only the fundamental mode. This equation describes the dynamics of a convective pattern near the sinusoidal pattern under the modulational noise. On the other hand, in the 2D system, when the roll aligns almost in the x direction, there still exists fluctuation of pattern alignment in the y direction. This can be taken into account as follows. Since the mode satisfying the wave number $k_x^2+k_y^2=k_0^2$ is most unstable, and $k_x=k_0+\delta k_x$, with δk_x being small, we get $2k_0\delta k_x+k_y^2=0$. So, by noting that the fluctuation scales as l_x and l_y respectively, in the x and y directions are related as $l_x\sim l_y^2$, the amplitude ψ defined via $w=\psi e^{ik_0x}+$c.c., obeys

$$\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\left[\lambda+f(t)+4k_0^2\left(\frac{\partial}{\partial x}-\frac{i}{2k_0}\frac{\partial^2}{\partial y^2}\right)^2\right]\psi-3|\psi|^2\psi. \tag{7}$$

Here the approximation $w^3\approx 3|\psi|^2\psi e^{ik_0x}+$c.c. has been used. This is an extension of the Newell-Whitehead equation in a simple, neutral fluid system [12] to the case with the fluctuation-modulated threshold.

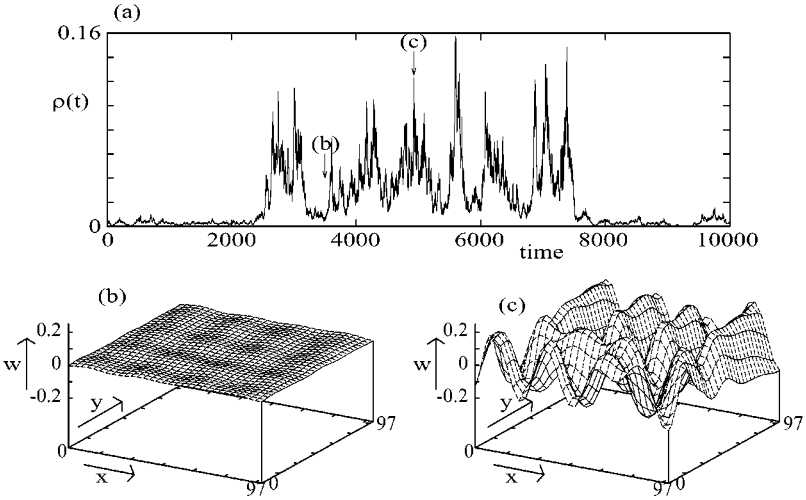

FIG. 1. (a) Temporal evolution of $\rho(t)$ obtained by numerically solving Eq. (3). The initial condition was chosen such that initially $w(\mathbf{r},0)$ at each lattice point, is randomly distributed close to zero. The parameter values are $D=10$ and $\gamma=50$ ($\Gamma_f=4\times10^{-3}$). (b) and (c) Spatial patterns at the times 3500 and 4927, respectively. These patterns correspond to the times denoted by the arrows in (a).

III. ON-OFF CONVECTION AND EFFECT OF THERMAL NOISE

Hereafter we consider the 2D system contained in a box with the linear scale L. The quantity $\rho(t)$ which measures the extent of the deviation from the planar alignment of director, is defined by

$$\rho(t)=\left\{\frac{1}{L^2}\int[w(\mathbf{r},t)]^2 d\mathbf{r}\right\}^{1/2}. \tag{8}$$

In other words, this quantity evaluates the intensity of the convective pattern. If there exists no convective motion, one obtains $\rho(t)=0$.

In order to carry out the numerical integration of Eq. (3) for the 2D system, the system is divided into $N\times N$ sites with the lattice spacing Δx. We will use the periodic boundary condition. The applied spatially uniform random force $f(t)$ is assumed to be generated by the Ornstein-Uhlenbeck process

$$\dot{f}(t)=-\gamma f(t)+R(t), \tag{9}$$

where $R(t)$ is the Gaussian white noise with the statistics

$$\langle R(t)\rangle=0,\ \langle R(t)R(0)\rangle=2D\delta(t) \tag{10}$$

with positive constants γ and D corresponding, respectively, to the inverse characteristic time of $f(t)$ and the intensity of $R(t)$. The noise intensity Γ_f of $f(t)$ is therefore given as

$$\Gamma_f=\frac{D}{\gamma^2}. \tag{11}$$

We numerically solved Eq. (3) with the quasispectral method, combining the Euler method for time integration. Namely, given $w(\mathbf{r},t)$, in terms of its Fourier transform $w_{\mathbf{k}}(t)$, Eq. (3) is numerically solved as

$$w_{\mathbf{k}}(t+\Delta t)=w_{\mathbf{k}}(t)+\{[\lambda_{\mathbf{k}}+f(t)]w_{\mathbf{k}}(t)-(w^3)_{\mathbf{k}}(t)\}\Delta t, \tag{12}$$

with the combination of solving Eq. (9) as

$$f(t)=\frac{[1-(\gamma/2)\Delta t]f(t-\Delta t)+\sqrt{2D\Delta t}G(t)}{1+(\gamma/2)\Delta t}, \tag{13}$$

where $G(t)$ is the Gaussian noise with the mean $\langle G(t)\rangle=0$ and the variance $\langle G(t)^2\rangle=1$. Carrying out the inverse Fourier transform of $w_{\mathbf{k}}(t+\Delta t)$, we obtain $w(\mathbf{r},t+\Delta t)$. Repeating this procedure, we can numerically integrate Eq. (3). Hereafter we use the parameter values $N=32$, $\lambda=2\times10^{-3}$, $k_0=0.25$, $\Delta x=\pi$, $\Delta t=5\times10^{-4}$, unless otherwise stated.

Figure 1(a) shows the temporal evolution of $\rho(t)$ obtained by numerically solving Eq. (3) and Figs. 1(b) and 1(c) are the spatial patterns, respectively, corresponding to the times denoted by the arrows in Fig. 1(a). One clearly observes that the temporal evolution of $\rho(t)$ shows an intermittency composed of laminar time intervals where no apparent pattern change is observed and a burst region where the spatial pattern with the wave number k_0 is observed. The pattern formation is observed in temporally highly localized regions. The temporal evolution, as shown in Fig. 1(a), is quite similar to the so called on-off intermittency [4–8]. In order to examine the statistics of the temporal evolution of $\rho(t)$, we will compare the statistical laws of on-off intermittency with those from Fig. 1(a). To do that, we first have to know the value of the noise intensity of local transverse expansion rate. This is calculated as follows. If $\rho(t)$ is sufficiently small, it obeys

$$\dot{\rho}(t)=[\lambda+f(t)]\rho(t)-\frac{1}{\rho(t)L^2}\times\int[(\nabla^2 w)^2-2k_0^2(\nabla w)^2+k_0^4 w^2]d\mathbf{r}. \tag{14}$$

We neglected the contribution from the nonlinear term w^3 in Eq. (3) because $\rho(t)$ is small enough, and used the periodic boundary condition. Since the most unstable mode right above the threshold ($\lambda>0$) has the wave number k_0, we make the approximation to replace $(\nabla^2 w)^2$ and $(\nabla w)^2$ in Eq. (14), respectively, by $k_0^4 w^2$ and $k_0^2 w^2$. This replacement leads to the approximate equation of motion

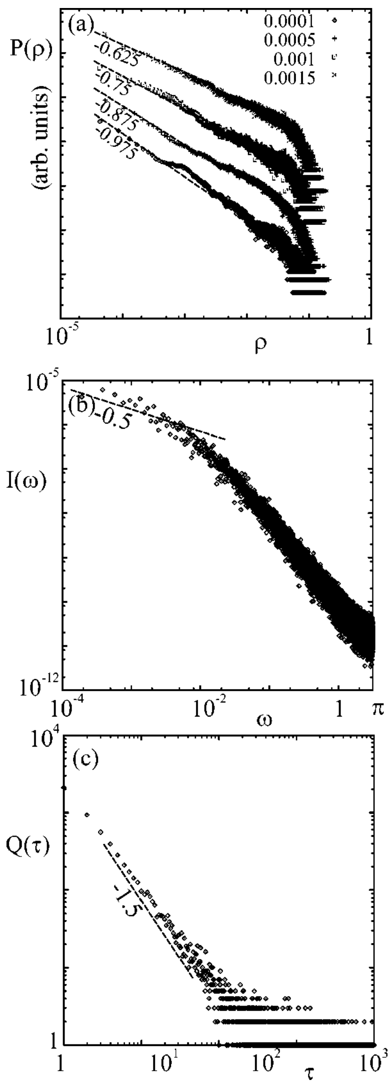

FIG. 2. Statistical laws of $\rho(t)$ numerically obtained: (a) probability density; (b) power spectrum; and (c) laminar duration distribution. For details see the text.

$$\dot{\rho}(t)=[\lambda+f(t)]\rho(t). \tag{15}$$

This equation has a particular solution $\rho=0$, which is consistent with the fact that Eq. (3) always has the same particular solution $w(\mathbf{r},t)=0$ for any t and $\mathbf{r}$. This equation is the same as the linearized stochastic multiplicative noise model, and therefore the extra exponent η in $P(\rho)$, the steady probability density for $\rho(t)$, is given by $\eta=\lambda/\Gamma_f$, as in Eq. (1).

Figures 2(a), 2(b), and 2(c) are comparisons with numerical results derived from the time series of $\rho(t)$ and the above theoretical results of on-off intermittency. Probability densities are obtained for $\lambda=10^{-4}$, 5×10^{-4}, 10^{-3}, and 1.5×10^{-3}, and $\Gamma_f=4\times10^{-3}$ ($\gamma=50,D=10$). Values of the exponent $(-1+\eta)$ in $P(\rho)\propto\rho^{-1+\eta}$, corresponding to several values of λ, were calculated by $\eta=\lambda/\Gamma_f$, and are denoted by $-0.625,-0.75$, etc., in Fig. 2(a). One finds good agreement with numerical results. In Fig. 2(b), the power law $\omega^{-(1/2)}$ is observed in a small ω region, although the region where it holds is not sufficiently wide. It is expected that if we use a more long-time series, the $\omega^{-(1/2)}$ region may become wider. In Fig. 2(c), the laminar duration distribution clearly shows the $\tau^{-(3/2)}$ law. Here the threshold ρ_{th} separating laminar and burst states is chosen as 0.01. The exponent 3/2 does not depend on the choice of ρ_{th} as long as it is small enough. One finds that the agreement of the present statistical characteristics, with those of on-off intermittency, is quite well. This numerically proves that the present intermittency in Fig. 1(a) is identical to the on-off intermittency. Analyzing the temporal evolution of the pattern intensity, we found that the pattern appears intermittently provided the planar alignment with $w\approx0$ becomes slightly unstable under the application of the multiplicative noise. We call this phenomenon *on-off convection*.

The next question concerns the temporal evolution of pattern. A primitive picture on the temporal evolution of pattern is as follows. Given initial condition, the system (3) approaches a steady state and forms an on-off convective pattern, whose wavelength is about $2\pi/k_0$ for small $\lambda(>0)$. Due to the multiplicative noise effect, this pattern disappears (the laminar region). The disappearance of pattern may destroy the memory of the details of the previous precise pattern. In a certain time, a new pattern would be suddenly generated, and this pattern might be different from the previous one. Figure 3 shows the numerical results of the temporal evolution of patterns. Figures 3(b) and 3(c) are patterns at the times denoted by the arrows in the temporal evolution of $\rho(t)$ of Fig. 3(a). Between the times corresponding to Figs. 3(b) and 3(c), there exist several laminar regions. The above picture on the pattern change leads to the conclusion that the patterns at these times are different. However, as shown in Figs. 3(b) and 3(c), the system almost keeps its initial pattern once it is created, and only the pattern intensity intermittently changes in time. This means that the pattern reserves its memory in a robust way. This fact can be understood as follows. When a pattern decays due to the noise effect after a pattern is formed, the equation of motion (3) can be approximated only by the linear term if $\rho(t)$ becomes quite weak. Even for small $\rho(t)$, the system still reserves its pattern $\{w(\mathbf{r})\}$, which again increases their local intensity of $\{w(\mathbf{r})\}$ in a spatially synchronized way when noise with positive $\{\lambda+f(t)\}$ is applied. Therefore, the pattern growth and decline sets in, in a synchronized way. This is the reason why the pattern form does not change in time, except for its intensity.

As shown above, the convective pattern does not change for the equation of motion (3). In real systems, the temporal evolution is affected by thermal noise. To take thermal noise into account, we consider the equation of motion

$$\frac{dw(\mathbf{r},t)}{dt}=[\lambda+f(t)-(\boldsymbol{\nabla}^2+k_0^2)^2]w-w^3+g(\mathbf{r},t), \tag{16}$$

where g represents the thermal noise and is supposed to be Gaussian white,

$$\langle g(\mathbf{r},t)\rangle=0,\quad \langle g(\mathbf{r},t)g(\mathbf{r}',t')\rangle=2\Gamma_g\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{17}$$

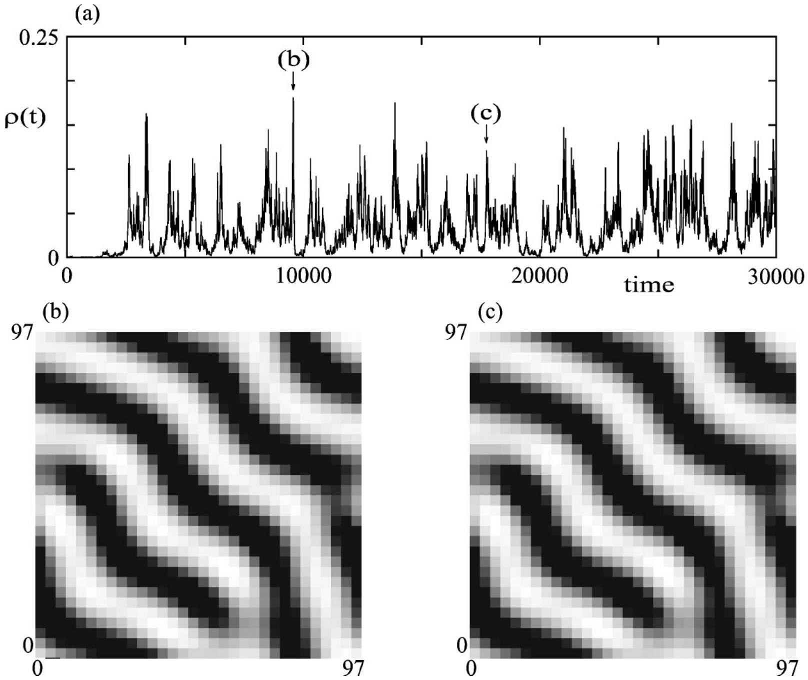

FIG. 3. (a) Temporal evolution of $\rho(t)$ without thermal noise for $\lambda=2\times10^{-3}$. (b) and (c) Spatial patterns of $w(\mathbf{r},t)$, respectively, at times $t=9568$ and 17 738 denoted by the arrows in (a).

The Γ_g evaluates the intensity of thermal noise. Equation (16) is numerically solved as

$$w(\mathbf{r},t+\Delta t)=w^0(\mathbf{r},t+\Delta t)+\sqrt{2\Gamma_g\frac{\Delta t}{(\Delta x)^2}}G(\mathbf{r},t), \tag{18}$$

where $w^0(\mathbf{r},t+\Delta t)$ is equivalent to $w(\mathbf{r},t+\Delta t)$ numerically obtained in the case without thermal noise, i.e., the inverse Fourier transform of Eq. (12). The $G(\mathbf{r},t)$ is the Gaussian noise with the vanishing mean and the variance

$$\langle G(\mathbf{r}_{jl},t_n)G(\mathbf{r}_{j'l'},t_{n'})\rangle=\delta_{jj'}\delta_{ll'}\delta_{nn'}, \tag{19}$$

where $\mathbf{r}_{jl}$ is the lattice position corresponding to the 2D lattice point index (j,l) and $t_n=t_0+n\Delta t$, $(n=0,1,2,\ldots)$. Figure 4(a) displays the temporal evolution of $\rho(t)$ and shows that the intermittency characteristic is still reserved as far as Γ_g is small. Figures 4(b) and 4(c) show the spatial patterns at times denoted in Fig. 4(a). These figures clearly show the spatial pattern after insertion of laminar states. This proves that the pattern change is not due to the threshold fluctuation but due to the existence of the additive noise corresponding to thermal noise.

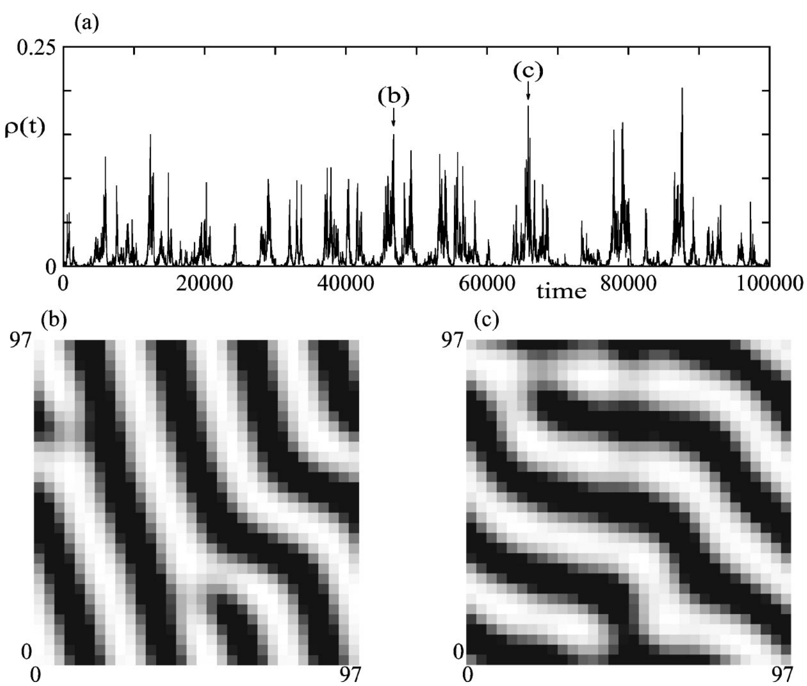

FIG. 4. (a) Temporal evolution of $\rho(t)$ with thermal noise for $\lambda=10^{-4}$ and $\Gamma_g=\pi^2\times10^{-8}$. (b) and (c) Spatial patterns of $w(\mathbf{r},t)$ at times $t=46\,745$ for (b) and 65 813 for (c) denoted by the arrows in (a).

IV. CONCLUDING REMARKS

In the present paper, we proposed a phenomenological stochastic model of instability predicted by Behn, Lange, and John and very recently experimentally found by John, Stannarius, and Behn (JSB) in EC in a nematic liquid crystal subject to a dichotomous stochastic electric field. The model is fundamentally the same as the Swift-Hohenberg equation but contains the modulation threshold.

Numerically solving the phenomenological model, we found that the statistical laws derived from the temporal evolution of the model dynamics, are the same as those known for the on-off intermittency reported in systems with small degrees of freedom. On the other hand, John *et al.*, observed that the temporal evolution after the instability of the planar alignment of the director is an intermittent characteristic, and found that the intermittency exhibits the third statistical law of on-off intermittency $Q(\tau)\sim\tau^{-(3/2)}$ for the laminar duration distribution. In order to identify that the intermittency they found was on-off intermittency, it required checking two other statistical laws.

Furthermore, by adding an additive noise, we studied the role of thermal noise. It was found that if thermal noise is absent, no practical pattern change is observed. For the experimental situation, thermal noise does not play a crucial role, and John *et al.*, also do not observe a meaningful change of pattern in the course of time, although the significant, on-off intermittent pattern change is observed [16]. However, no experimental study on pattern change has been carried out yet. A numerical and theoretical study as well as an experimental study along this line is also desirable.

Furthermore, we carried out the numerical integration of Eq. (3) by applying the dichotomous noise $f(t)$ with the vanishing mean and $\langle f(t)f(0)\rangle = f_0^2 e^{-\gamma|t|}$ instead of the Gaussian noise obeying the Ornstein-Uhlenbeck process, where $\gamma>0$ is the decay rate of correlation, and found that the qualitative results of the statistics do not change. This fact implies that the present on-off convection is free from details of applied noise, and supports the universality of the onset of on-off convection. In this sense, the present model deserves a simple nonlinear stochastic model of electrohydrodynamic convection, theoretically and experimentally studied by Behn and his co-workers.

JSB reported the experimental phase diagram for conductive and convective regions in the $\nu(\equiv\gamma)-U(\equiv Ed)$ plane, where γ and E are, respectively, the decay rate of the correlation function of the dichotomous noise and its intensity, and d is the thickness of the electrodes. One should note that the present parameters, e.g., λ and $f(t)$, do not directly correspond to the $E-E_c$ and $E(t)$, respectively, where $E(t)$ is the applied dichotomous noise and E_c is its critical value for a given ν. Therefore, although our model can describe the quantitative statistics of on-off convection, it is not suitable for the explanation of the experimental phase diagram.

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[1] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[2] P. Manneville and Y. Pomeau, Physica D **1**, 219 (1980).
[3] H. G. Schuster, *Deterministic Chaos* (VCH, Weinheim, 1988).
[4] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 918 (1985); **75**, 1087 (1986).
[5] T. Yamada and H. Fujisaka, Prog. Theor. Phys. **76**, 582 (1986); **84**, 824 (1990); H. Fujisaka and T. Yamada, *ibid.* **77**, 1045 (1987); **90**, 529 (1993); H. Suetani and T. Horita, Phys. Rev. E **60**, 422 (1999); H. Fujisaka, H. Suetani, and T. Watanabe, Prog. Theor. Phys. Suppl. No. 139, 70 (2000); H. Fujisaka, S. Matsushita, and T. Yamada, J. Phys. A **30**, 5697 (1997); S. Miyazaki, J. Phys. Soc. Jpn. **69**, 2719 (2000).
[6] N. Platt, E.A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. E **49**, 1140 (1994).
[7] T. Yamada, K. Fukushima, and T. Yazaki, Prog. Theor. Phys. Suppl. No. 99, 120 (1989); E. Ott and J.C. Sommerer, Phys. Lett. A **188**, 39 (1994); Y.C. Lai and C. Grebogi, Phys. Rev. E **52**, R3313 (1995); A. Čenys, A. Namajunas, A. Tamserius, and T. Schneider, Phys. Lett. A **213**, 259 (1996); S.C. Venkataramani, T.M. Antonsen, Jr., E. Ott, and J.C. Sommerer, Physica D **96**, 66 (1996); Y.-C. Lai, Phys. Rev. E **53**, R4267 (1996); **54**, 321 (1996); T. Harada, H. Hata, and H. Fujisaka, J. Phys. A **32**, 1557 (1999).
[8] F. Rödelsperger, A. Cenys, and H. Benner, Phys. Rev. Lett. **75**, 2594 (1995); H. Fujisaka, K. Ouchi, H. Hata, B. Masaoka, and S. Miyazaki, Physica D **114**, 237 (1998); J. Becker, F. Rödelsperger, Th. Weyrauch, H. Benner, W. Just, and A. Čenys, Phys. Rev. E **59**, 1622 (1999).
[9] U. Behn, A. Lange, and Th. John, Phys. Rev. E **58**, 2047 (1998).
[10] *Pattern Formation in Liquid Crystals*, edited by A. Buka and L. Kramer (Springer-Verlag, Berlin, 1996); W. Pesch and U. Behn, in *Evolution of Spontaneous Structures in Dissipative Continuous Systems*, edited by F. H. Busse and S. C. Müller (Springer, Heidelberg, 1998).
[11] Th. John, R. Stannarius, and U. Behn, Phys. Rev. Lett. **83**, 749 (1999).
[12] A.C. Newell and J.A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[13] J. Swift and P.C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[14] P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic, New York, 1990).
[15] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[16] U. Behn, R. Stannarius, and Th. John (private communication).